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# Quantisation of soliton solution of a generalised discrete non-linear equation

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**Abstract.** Soliton solution of a discrete non-linear equation of which the KdV, Volterra and Toda equation are particular cases, have been quantised by the semi-classical method of Dashen *et al.* The primary feature of the system is the emergence of the same stability angle in all three cases. The discrete energy levels of the quantised system were found to depend on the characteristic constants of the individual non-linear equations.

## 1. Introduction

In recent years soliton solutions of non-linear equations have been investigated quite exhaustively for constructing models of quark confinement (Joos 1975) and of hadrons with finite extension (Popov 1977). The most pertinent step in the above mentioned programmes is the quantisation of the classical lump. In general there exist two distinct methods of procedure for the system to be quantised; one is the general method after Dirac (1950) and Faddeev (Faddeev and Takhatajan 1974) which takes care of all possible constraints on the system; and the other is the semi-classical method of Dashen *et al* (1974, 1975) which is suitable for periodic solutions. In this work we have applied the WKB approach to a discrete system for exacting the energy level scheme.

## 2. Formulation

The non-linear equation under consideration reads:

$$\begin{aligned} \dot{u}_n &= [\alpha^{-1}/4(\beta_1^2-1)u_n^2+\beta_1 u_n+\alpha](v_{n-1/2}-v_{n+1/2}) \\ \dot{v}_n &= [\alpha/4(\beta_2^2-1)v_n^2+\beta_2 v_n+\alpha^{-1}](u_{n-1/2}-u_{n+1/2}) \end{aligned} \tag{1}$$

where $\alpha, \beta_1, \beta_2$, are three constants whose special values yield the three known equations as particular cases. The one- and two-soliton solutions of equation (1) have been obtained by Hirota and Satsuke (1976). The periodic one-soliton solution reads

$$u_n = 2\omega \sinh b/[\cosh b + \cos(2\omega t - 2kn)] \tag{2}$$

with a similar expression for $v_n$. Equation (2) describes a soliton propagating with a speed $\sinh P/P$. Here $\omega$ is the frequency, $k$ is the wavevector of the soliton and $b$ is a constant given by

$$\exp 2b = \frac{\epsilon a^{-1}\hat{\beta}_1+\hat{\beta}_2\cos k+\sin k}{\epsilon a^{-1}\hat{\beta}_1+\hat{\beta}_2\cos k-\sin k} \qquad \epsilon=\pm 1.$$

  

The first step in the quantisation procedure is the calculation of the stability angles defined by Faddeev and Takhatajan (1974). To do this we set in equation (1)

$$u_n = u_n(1s) + \delta u_n$$
$$v_n = v_n(1s) + \delta v_n \tag{3}$$

where $u_n(1s)$ and $v_n(1s)$ are one-soliton solutions. This gives the linearised equations:

$$\begin{aligned}\delta \dot{u}_n &= [\alpha^{-1}/2(\beta_1^2 - 1)u_n + \beta_1]_0(v_{n-1/2} - v_{n+1/2})\,\delta u_n \\ &\quad + \{\beta_1 u_n + \alpha + (\alpha^{-1}/4)(\beta_1^2 - 1)\}_0(\delta v_{n-1/2} - \delta v_{n+1/2}) \\ \delta \dot{v}_n &= [\alpha/2(\beta_2^2 - 1)v_n + \beta_2]_0(u_{n-1/2} - u_{n+1/2})\,\delta v_n \\ &\quad + [(\alpha/4)(\beta_2^2 - 1)v_n^2 + \beta_2 v_n + \alpha^{-1}](\delta u_{n-1/2} - \delta u_{n+1/2}).\end{aligned} \tag{4}$$

The stability angles are obtained by the requirement that

$$\delta u_n(t+\tau) = e^{\omega_n(\tau)}\,\delta u_n(t)$$
$$\delta v_n(t+\tau) = e^{\omega'_n(\tau)}\,\delta v_n(t)$$

$\omega_n$, $\omega'_n$ representing the stability angles. But as the system (1) is completely integrable, one need not solve equation (4) and instead one can take recourse to the method used by Dashen. That is, we examine a particular two-soliton solution for $u_n$ and $v_n$, each of which contains the usual singlet and another one with different internal period and amplitude. We then examine the behaviour of the system as the amplitude of the second soliton tends to zero. The two-soliton solution following Hirota is given by

$$u_n = \lg(f'_n/f_n); \qquad v_n = \lg(g'_n/g_n); \qquad \eta_i = \Omega_i t - P_i \eta - \eta_i^0;$$
$$\Omega_i = \epsilon_i \sinh P_i; \qquad \epsilon_i = \pm 1$$

$$f_n = \sum_{\mu=0,1} \exp\left(\sum_{\mu_i=1}^{2} \mu_i(2\eta_i + \phi_i) + \sum A_{ij}\mu_i\mu_j\right)$$

$$g_n = \sum_{\mu=0,1} \exp\left(\sum_{\mu_i=1}^{2} \mu_i(2\eta_i + \psi_i) + \sum A_{ij}\mu_i\mu_j\right)$$

$$f'_n = \sum_{\mu=0,1} \exp\left(\sum_{\mu_i=1}^{2} \mu_i(2\eta_i + \phi'_i) + \sum A_{ij}\mu_i\mu_j\right)$$

$$g'_n = \sum_{\mu=0,1} \exp\left(\sum_{u_j=1}^{2} \mu_j(2\eta_i + \psi'_j) + \sum A_{ij}\mu_i\mu_j\right)$$

$$\exp A_{ij} = \left(\frac{\epsilon_i \exp P_i - \epsilon_j \exp P_j}{1 - \epsilon_i\epsilon_j \exp(P_i + P_j)}\right)^2$$

$$\exp \phi_i = \epsilon_i \alpha^{-1}\beta_1 + \beta_2 \cosh P_i - \sinh P_i$$

$$\exp \phi'_i = \epsilon_i \alpha^{-1}\beta_1 + \beta_2 \cosh P_i + \sinh P_i$$

$$\exp \psi_i = \epsilon_i \alpha^{-1}(\epsilon_i \alpha\beta_2 + \beta_1 \cosh P_i - \sinh P_i)$$

$$\exp \psi'_i = \epsilon_i \alpha^{-1}(\epsilon_i \alpha\beta_2 + \beta_1 \cosh P_i + \sinh P_i)$$

and where $P_i$, $\eta_i^0$ are real constants. To analyse the behaviour of the system as the amplitude of the second soliton goes to zero, we set $e^{\gamma_1} = C_1$, $e^{\gamma_2} = C_2$ as the amplitude of the two. We then expand the expression for two solitons from (4) in $C_2$ and collect terms linear in $C_2$ which yields the solution of the linearised equation (4) as

$$\delta u_n = 2\{e^{\phi_2}B - e^{\phi_2'}A\} \exp\{i(\sin k_2 t + k_2 n)\}$$

with $A$ and $B$ given by

$$A = \{\sin k_2 + (\sin k_1 + \sin k_2)\, e^{2\bar{\eta}_1 + \phi_1' + A_{12}}\}/(1 + e^{2\bar{\eta}_1 + \phi_1'})$$

$$B = \{\sin k + (\sin k + \sin k)\, e^{2\eta_1 + \phi_1 + A_{12}}\}/(1 + e^{2\bar{\eta}_1 + \phi_1})$$

$$\bar{\eta}_1 = \Omega_1 t - P_1 n.$$

To quantise the solution we need the classical energy and action. To compute these we note that the Lagrangian of the system is

$$L = \tfrac{1}{2}(u_n\dot{u}_n + v_n\dot{v}_n) - [\alpha^{-1}(\beta_1^2 - 1)(u_n^3/12) + \beta_1(u_n^2/2) + \alpha u_n](v_{n-1/2} - v_{n+1/2})$$
$$- [\alpha(\beta_2^2 - 1)(v_n^3/12) + \beta_2(v_n^2/2) + \alpha^{-1} v_n](v_{n-1/2} - u_{n+1/2}). \tag{8}$$

Substituting the classical solution (2) into (8) and integrating over one period we obtain

$$S_{\mathrm{cl}} = \int_0^\tau 2\, \mathrm{d}t - = \frac{C}{\tau} + \frac{A}{\tau^2} + \frac{B}{\tau^3} \tag{9}$$

where $A$, $B$, $C$ are given in terms of the Fourier expansion coefficients of the soliton solutions and are of the form

$$A = 2[\sinh^2 b \Sigma a_n^2 + \Sigma b_n^2 \sinh^2 f]$$
$$+ 2\pi\beta_1 \sinh f \sinh^2 b\{\Sigma a_n a_l a_{n+1} + 2a_n a_{-l} a_{l+n} + a_l a_n a_{-(n+l)}\} \tag{10}$$

where

$$u_n = \sum a_l \cos[l(\omega t - kn)] \tag{11}$$

with $a_l = [\alpha l/(\alpha - \beta)]$, $\alpha, \beta$ being roots of the equation $z^2 + 2z \sinh b + 1 = 0$ with a similar expression for $v_n$. Also, from the expression of the Hamiltonian, using energy conservation we get

$$E_{\mathrm{cl}} = \frac{C}{\tau^2} + \frac{A}{\tau^3} + \frac{B}{\tau^4}. \tag{12}$$

The last stage in the quantisation is to evaluate $\mathrm{Tr}(H - E)^{-1}$. We note in the usual manner that

$$Gc(E) = \mathrm{Tr}(H - E)^{-1} = (i/\hbar)\, \mathrm{Tr} \int_0^\infty \exp\left[\frac{it}{\hbar}(H - E)\right]$$
$$= \frac{i}{\hbar}\sqrt{\frac{-i}{2\pi\hbar}} \sum\sum \int \mathrm{d}\tau \sqrt{n\tau} \left|\frac{\partial E}{\partial \tau}\right|^{1/2} \exp\left[\frac{in}{\hbar}(S_{\mathrm{cl}}(\tau) - \xi_{nk} + E\tau)\right]. \tag{13}$$

Integrating by the method of steepest descent we get

$$\frac{\partial S_{\mathrm{cl}}}{\partial \tau} = -E_{\mathrm{cl}}(\tau) = E + (\mathrm{d}/\mathrm{d}\tau)\xi_{nk}(\tau). \tag{14}$$

Equation (14) solves for $\tau$ as a function of energy. Essentially equation (14) is a biquadratic equation in $\tau$ and can be depicted as

$$\tau'^4 \beta + \tau'^3 A + \tau'^2 C = E - \xi'_{nk}. \tag{15}$$

Furthermore we have from the consideration that the Green function should have pole

$$W_{\{nk\}}(E) = \underset{m=0,1,2,3,\dots}{2\pi\hbar m} \tag{16}$$

where $W_{\{nk\}}$ is defined as

$$W_{\{nk\}} = S_{\rm cl}(\tau(E)) + E\tau - \xi_{nk}(\tau(E)). \tag{17}$$

When the value of $(E)$ from equation (14) is substituted in (16) we get an expression for $H$, the quantised energy of the system.

## 3. Discussion

The most interesting observation that emerges from the above discussions is that the stability angles $\omega_k(\tau)$ are independent of the constants and $\alpha$. That is all the three non-linear equations KdV, Toda Lattice and the Volterra system have the same stability angles but since the classical action and energy $S_{\rm cl}$, $E_{\rm cl}$ depend on $\beta_1$, $\beta_2$ and $\alpha$ through the constants $A$, $B$, $C$ the quantised energy levels $E$ obtained, equation (17) becomes completely different in each case.

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